AN ALGORITHMIC SOLUTION FOR FRICTIONAL CONTACT PROBLEMS SUBJECTED TO A TWISTING MOMENT

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An efficient numerical procedure is devised and applied for the frictional contact problems subjected to a normal load and a twisting moment. The traction distribution and the region of micro-slip on any shape of contact area can be effectively found by iteratively using a modified linear programming technique. The compliance of the contact system is also evaluated. The numerical solution obtained by the suggested procedure agrees very well with Lubkin's theory for the circular contact area. The algorithmic solution for disconnected contact areas on semi-infinite bodies is presented to illustrate generality and effectiveness of the proposed numerical procedure.

Key Words : Frictional Contact, Torsion, Surface Traction Distribution

1. INTRODUCTION

A small torsional couple is applied across the contact surface of two elastic bodies which have been pressed together by a force perpendicular to their common tangent plane(Fig. 1.). From a solution by H. Neuber which pertained to a hyperbolic groove in a twisted shafte, R.D. Mindlin obtained solutions for the circular and the elliptic contact area under the assumption of no slip on the interface(Mindlin, 1949). The torsional compliance was given in terms of complete elliptic integrals and the resultant tangential traction was found to rise from zero at the center to infinity at the
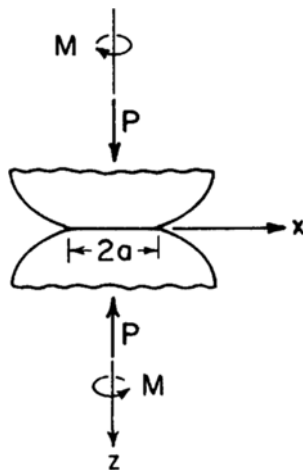

Fig. 1 Contact of spherical bodies subjected to a twisting moment

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edge of the contact surface, which is physically unattainable.

J.L. Lubkin allowed slip in circular contact by assuming that the tangential component of traction is equal to the product of a constant friction coefficient and the Hertzian pressure for the region over which slip occurs(Lubkin, 1951). He obtained a continuous expression for the traction, a formula relating the radial depth of penetration of slip to the magnitude of the applied moment and a relationship between the moment and the rigid body rotation.

H. Deresiewicz extended the Lubkin solution to the contact of elastic sphere under an oscillating torsional couple (Deresiewicz, 1954).

M. Hetényi and H. McDonald treated only the limiting case where the gross torsional movement impends for the contact of elastic spheres(Hetényi and Macdonald, 1958).

All of the analytical solutions for the firictional contact problem, however, are restricted to the case of the circular contact area.

The purpose of this study is to formulate a general and effective procedure for evaluating the traction distribution, the localized microslip region and the compliance on any shape of contact area formed by a normal load under a twisting moment lower than that necessary to cause gross spin. The suggested procedure utilizes a modified linear programming technique which was introduced to solve the frictionless contact problem subjected to a normal load(Conry and Seireg, 1971).

2. BOUNDARY CONDITIONS

Within the limits of the small strain and small rotation theory, Mindlin presumed that there is no change in the normal component of traction across the contact surface and that there is no relative displacement on the contact surface under the assumption of no slip when a twisting moment is superposed on a normal load(Mindlin, 1949). The problem is thus reduced to a problem of the plane with the following

boundary conditions.

$$U_\theta = r\gamma, \ U_r = \sigma_z = 0 \text{ inside the contact boundary} \quad (1)$$
$$\tau_{zr} = \tau_{z\theta} = \sigma_z = 0 \quad \text{outside the contact boundary} \quad (2)$$
$$\lim_{r,z \to \infty} (U_r, U_\theta, U_z) = 0 \quad (3)$$

where

U_r, U_θ, U_z : Cylindrical coordinate components of displacement

$\tau_{zr}, \tau_{z\theta}, \sigma_z$: Cylindrical coordinate components of traction on the plane $z = 0$

r : Radial distance from the center of rotation

γ : Angle of rigid rotation of the entire contact surface relative to distant lines in the solid

Because the tangential component of traction becomes infinite at the edge of the contact surface(Mindlin, 1949), slip must be expected to occur, starting at the goundary and presumably progressing inward.

To accomodate the evident slip, we retain Eqs.(2) and (3) but must modify Eq.(1). In the slip region, the tangential component of traction τ is assumed to take the value of the product of a constant friction coefficient f and the normal contact pressure p, following Cattaneo(Cattaneo, 1983). This proportionality has been corroborated by photoelastic measurements(Hetényi and MacDonald, 1958). In the no-slip region, the rigid rotation γ remains unaffected and the surface traction is required to remain finite. the modified boundary conditions for the contact boundary can be expressed as

$$U_\theta = r\gamma, \ U_r = \sigma_z = 0 \quad \text{in the no-slip region} \quad (4)$$
$$\tau = fp, \ \sigma_z = 0 \text{ in the slip region} \quad (5)$$

3. PROPOSED ALGORITHM

3.1 Pre-Processor
The contact area is discretized into a finite number of rectangular grid elements of equal size. The tangential force and displacement at the center of each element are assumed to represent the distributed tangential components of traction and displacement over the finite area of each element respectively.

To linearize this nonlinear contact problem, the directions of discrete tangential forces are assumed to be oriented so as to make the direction of discrete tangential displacements circumferential at the grid points. A pre-processor determines these directions by enforcing the discrete tangential displacement to be in circumferential direction.

3.2 Problem Formulation with Specified Center of Rotation
The condition for compatibility of deformation can be stated as follows:

$$(U_\theta)_k + (Y_1)_k = r_k\gamma, \ k=1,2,\cdots,N \quad (6)$$

where

N : Number of discretized grid elements

$(U_\theta)_k$: Discretized displacement in circumferential direction at a point k

r_k : Radial distance of a point k from the center of rotation

γ : Angle of rigid rotation of the contact system

$(Y_1)_k$: Slack variable for deformation at a point k

$(Y_1)_k = 0$ in the no-slip region
$(Y_1)_k > 0$ in the slip region

The constraints on the traction force values can also be stated as:

$$F_k + (Y_2)_k = fP_k, \ k=1,2,\cdots,N \quad (7)$$

where

F_k : Discretized traction force in the direction determined by the pre-processor at a point k

f : Coefficient of friction

P_k : Discretized normal force at a point k

$(Y_2)_k$: Slack variable for the traction force at a point k

$(Y_2)_k > 0$ in the no-slip region
$(Y_2)_k = $ in the slip region

The codition for moment equilibrium is expressed as:

$$\sum_{k=1}^{N} (\hat{\theta}_k \cdot \hat{\phi}_k) r_k F_k = M \quad (8)$$

where

$\hat{\theta}_k$: Unit vector in circumferential direction at point k

$\hat{\phi}_k$: Unit vector in the direction of the traction force at a point k

M : Applied twisting moment

Since the point k must be either in the no-slip region or in the slip region, the following complementary condition should be satisfied.

$$\text{either } (Y_1)_k = 0 \text{ or } (Y_2)_k = 0 \quad (9)$$

Assuming elastic contact, the discretized elastic deformation at a point k can be presented in terms of the discretized traction forces as follows:

$$(U_\theta)_k = \sum_{j=1}^{N} s_{kj} F_j \quad (10)$$

where

S_{kj} : Elastic deformation in circumferential direction at a point k due to a unit force in the direction predetermined by the pre-processor at a point j

Incorporating the condition for the compatibility of deformation, the constraints on the traction force values, the condition for moment equilibrium, the complementary condition and the non-negativity conditions, the discretized contact problem can be formulated as follows:

Find$(\bar{F}, \bar{Y}_1, \bar{Y}_2, \gamma)$ which satisfies the following constraints

$$S\bar{F} + I\bar{Y}_1 = \bar{r}\gamma$$
$$I\bar{F} + I\bar{Y}_2 = f\bar{p}$$

$$\sum_{k=1}^{N} (\hat{\theta}_k \cdot \hat{\phi}_k) r_k F_k = M \qquad (11)$$

$$Y_{1k} = 0 \text{ or } Y_{2k} = 0, \qquad k = 1, \dots, N$$

$$F_k \geq 0, Y_{1k} \geq 0, Y_{2k} \geq 0, \gamma \geq 0, k = 1, \dots, N$$

where

S : N by N matrix of influence coefficients

I : N by N indentity matrix

$\bar{F}$: Vector of tangential forces in the directions pre-determined by the suggested pre-processor

$\bar{Y}_1$: Vector of slack variables for tangential displacements

$\bar{Y}_2$: Vector of slack variables for tangential forces

$\bar{P}$: Vector of normal forces

$\bar{r}$: Vector of radial distances from the specified center of rotation

Notice that all the variables are non-negative and all the constraints except the complementary condition are linear in the formulation. In other words, the problem can be solved using the powerful linear programming technique if the complementary condition can be incorporated in the technique. This complementary condition can be accomodated by modifying the entry rule of the present linear programming technique(Conry and Seireg, 1971). The above formulation is now restated in a form suitable for a modified linear programming as follows:

Minimize $\qquad \sum_{i=1}^{2N+1} z_i$

subject to $\qquad S\bar{F} + I\bar{Y}_1 - \bar{r}\gamma + I\bar{Z}_1 = 0$

$$I\bar{F} + I\bar{Y}_2 + I\bar{Z}_2 = f\bar{p} \qquad (12)$$

$$\sum_{k=1}^{N} (\hat{\theta}_k \cdot \hat{\phi}_k) r_k F_k + z_{2N+1} = M$$

$$Y_{1k} = 0 \text{ or } Y_{2k} = 0, \qquad k = 1, \dots, N$$

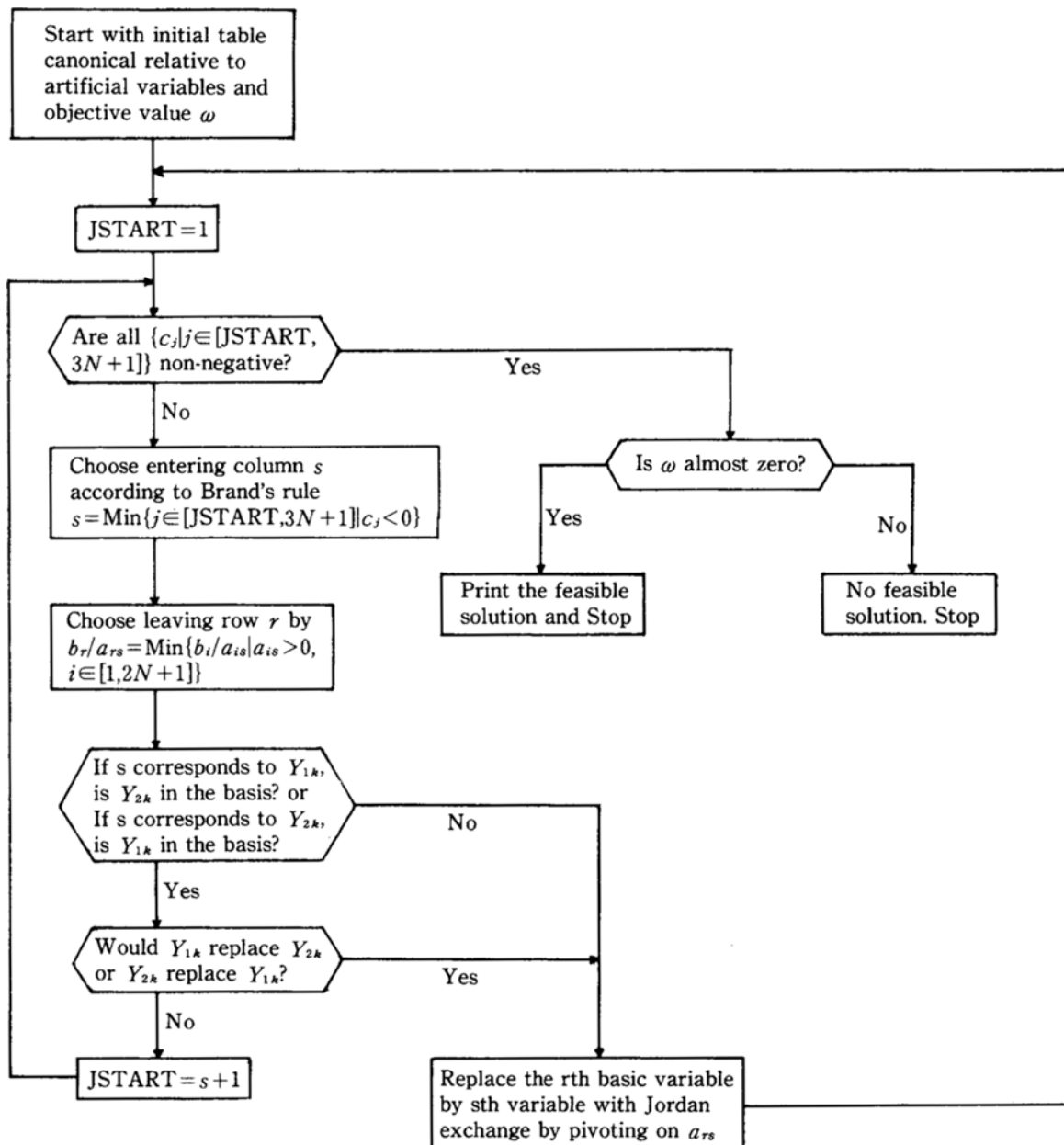$$F_k \geq 0, Y_{1k} \geq 0, Y_{2k} \geq 0, \gamma \geq 0, k = 1, \dots, N$$

Fig. 2 Flow chart for the modified linear programming

$$z_i \geqq 0 \qquad\qquad i=1,\ldots, 2N+1$$

where

$\bar{z}_1$: Vector of the first N artificial variables for the compatibility condition of deformation

$\bar{z}_2$: Vector of the next N artificial varibles for the constraints on the values of tangential forces

z_{2N+1} : Artificial variable for the moment equlibrium equation

The flow chart for the algorithm utilizing the modified linear programming is shown in Fig. 2.

3.3 Iterative Procedure

For the case in which the center of rotation is not easily found by inspection, e.g. the case of asymmetry, an iterative procedure is used to obtain the center of rotation (x_c, y_c) with respect to the centroid and the corresponding traction distribution, the local micro-slip region and the angle of rigid rotation. Since there is no applied tangential force, the sum of the discretized tangential forces, denoted the residual force here, should be zero when the real center of rotation is found. It is the exiting criterion of the following iterative procedure.

(1) Guess the first two centers of rotation and find the two corresponding residual forces at the first two iteration.

(2) Find the next center of rotation by the linear interpolation scheme.

(3) Determine the direction of discretized tangential forces by the pre-processor.

(4) Obtain the tangential traction distribution using the modified linear programming technique.

(5) Calculate the residual force.

(6) If the residual force is negligible, the solution is found. Otherwise, go to (2) and repeat until the solution is found.

4. FORMULATION USING NONLINEAR PROGRAMMING

To linearize the nonlinear contact problem for computational efficiency, the following two schemes have been employed in the proposed algorithm.

(1) The directions of the discretized tangential forces are predetermined about the specified center of rotation at each iteration under the implied assumption that the directions will not deviate much for those with slip allowed from those with no slip.

(2) The actual center of rotation is found by the iterative procedure described in 3.3.

The same problem is formulated by a system of nonlinear equations without any linearization. The rectangular components of the discretized tangential forces, the angle of rigid rotation and the center of rotation are obtained by solving the system of nonlinear equations once and will be compared with the solution found by using the proposed algorithm.

The condition for compatibility of deformation can be stated as:

$$|u_k| + (Y_u)_k = |-(y_k - y_c)\gamma|, \quad k=1,2,\ldots,N \qquad (13)$$

$$|v_k| + (Y_v)_k = |(x_k - x_c)\gamma|, \qquad k=1,2,\ldots,N \qquad (14)$$

where

u_k, v_k : Rectangular components of the tangential displacement at a point k

x_k, y_k : Rectangular components of the position of a point k

x_c, y_c : Rectangular components of the center of rotatioin

$(Y_u)_k, (Y_v)_k$: Non-negative slack variables for the displacements at a point k

     $(Y_u)_k = (Y_v)_k = 0$ in the no-slip region

     $(Y_u)_k, (Y_v)_k > 0$ in the slip region

The constraints on the values of tangential forces can also be stated as:

$$(X_k^2 + Y_k^2)^{1/2} + (Y_F)_k = fP_k \qquad (15)$$

where

X_k, Y_k : Rectangular components of the tangential force at a point k

$(Y_F)_k$: Non-negative slack variable for the force at a point k

     $(Y_F)_k > 0$ in the no-slip region

     $(Y_F)_k = 0$ in the slip region

The condition for the equlibrium can be expressed as:

$$\sum_{k=1}^{N} X_k = 0 \qquad \sum_{k=1}^{N} Y_k = 0$$

$$\sum_{k=1}^{N} [-(y_k - y_c)X_k + (x_k - x_c)Y_k] = M \qquad (16)$$

The complementary condition can be written as:

$$(Y_u)_k = (Y_v)_k = 0 \text{ or } (Y_F)_k = 0, \quad k=1,2,\ldots,N \qquad (17)$$

Assuming elastic contact, the deformations at a point k can be presented in terms of the forces as follows:

$$u_k = \sum_{j=1}^{N} (a_{kj}X_j + c_{kj}Y_j)$$

$$v_k = \sum_{j=1}^{N} (c_{kj}X_j + b_{kj}Y_j) \qquad (18)$$

where

a_{kj} : Deformation in the x-direction at a point k due to a unit force in the x-direction at a point j

b_{kj} : Deformation in the y-direction at a point k due to a unit force in the y-direction at a point j

c_{kj} : Either deformation in the x-direction at a point k due to a unit force in the y-direction at a point j or deformation in the y-direction at a point k due to a unit force in the x-direction at a point j

This system of non-linear equations with inequality constraints and the complementary condition can be greatly reduced into the equivalent system of equations without any inequality constraints by using the following lemma:

$$a = b_+ \rightleftarrows a \geqq 0, \quad a - b \geqq 0, \quad a(a-b) = 0$$

where   (19)

$$b_+ = \begin{cases} b \text{ if } b \geqq 0 \\ 0 \text{ if } b < 0 \end{cases}$$

To circumvent the difficulty of finding a set of solutions for a system of highly nonlinear equations, the formulation suitable for obtaining the solution $(\bar{X}, \bar{Y}, \gamma, x_c, y_c)$ using the nonlinear programming technique can be stated as follows:

Minimize $\sum_{i=1}^{2N+3} z_i^2$

subject to
$$[\{|-(y_k-y_c)\gamma|-|\sum_{j=1}^{N}(a_{kj}X_j+c_{kj}Y_j)|\}-\{fP_k$$
$$-(X_k^2+Y_k^2)^{1/2}\}]_+ -[|-(y_k-y_c)\gamma|-|\sum_{j=1}^{N}(a_{kj}X_j$$
$$+c_{kj}Y_j)|]+z_k=0, \quad k=1,\cdots,N$$

$$[\{|(x_k-x_c)\gamma|-|\sum_{j=1}^{N}(c_{kj}X_j+b_{kj}Y_j)|\}-\{fP_k-(X_k^2$$
$$+Y_k^2)^{1/2}\}]_+ -[|(x_k-x_c)\gamma|-|\sum_{j=1}^{N}(c_{kj}X_j+b_{kj}Y_j)|]$$
$$+z_{N+k}=0, \quad k=1,\cdots,N \qquad (20)$$

$$\sum_{k=1}^{N} X_k+z_{2N+1}=0$$
$$\sum_{k=1}^{N} Y_k+z_{2N+2}=0$$
$$\sum_{k=1}^{N}[-(y_k-y_c)X_k+(x_k-x_c)Y_k]+z_{2N+3}=M$$

5. COMPARISON BETWEEN THE PROPOSED INERATIVE PROCEDURE AND NON -LINEAR FORMULATION

At this point, the reasons for developing the iterative procedure utilizing the pre-processor and the modified linear formulation instead of using the complete non-linear formulation will be discussed.

First of all, the iterative procedure is guaranteed to yield the solution, while the non-linear formulation may or may not give the solution. Morever, the solution by non-linear formulation could be just one of many local solutions.

Secondly, the iterative procedure does not need the initial guess while the non-linear formulation needs a very good initial guess on all the variables.

Thirdly, the algorithm for the iterative procedure is definite and concise. Its codes can be easily written by the user while the effective algorithm should be provided for the non-linear formulation.

Finally, the iterative procedure is much more efficient than the non-linear formulation in terms of time, if a good initial guess cannot be given for the non-linear optimization. I such a case, the search may progress for a very long time.

6. ILLUSTRATIVE EXAMPLES

6.1 Circular Hertzian Contact

The frictional contact between a steel sphere($E=207$ GPa, $\nu=0.3$) with a radius of 25mm on a steel sphere of the same radius is selected as an example for comparison with the analytical solution by Lubkin(Lubkin, 1951). According to the Hertz theory, the applied normal load of $9600N$ forms the circular contact area of radius a and the contact pressure distribution p, the magnitudes of which are as follows(Johnson, 1985):

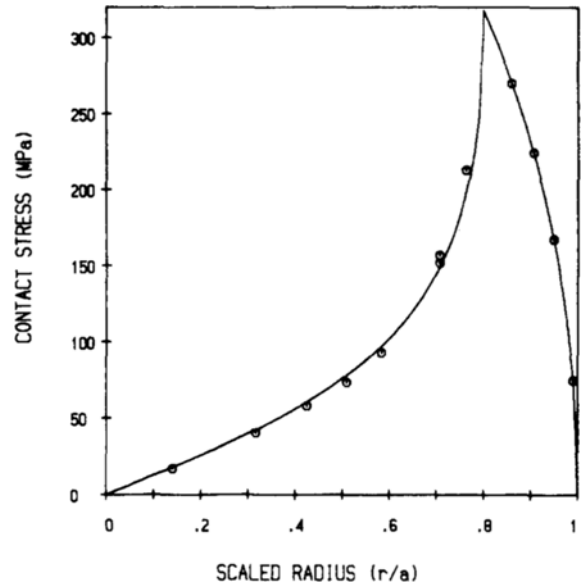$$a=0.93\text{mm}, \quad p=5310[1-(r/a)^2]^{1/2}MPa$$

Fig. 3   Contact stresses on clrcular contact area

A grid with 80 square elements is used to discretize the circular contact area. The following values are given:

$$M=0.3N\cdot m, \quad f=0.1$$

The influence coefficients are calculated by using the solution for a semi-infinite body subjected to a tangential force since the radius of the circular contact area is small in comparison with the radii of the spheres in contact(Love, 1944).

A comparison between Lubkin's theory(solid line) and the numerical results(symbol o) is plotted in Fig. 3, and good agreement can be seen. The rigid body rotation(0.10641×10^{-2} rad) is also found to compare favorably with Lubkin's theory
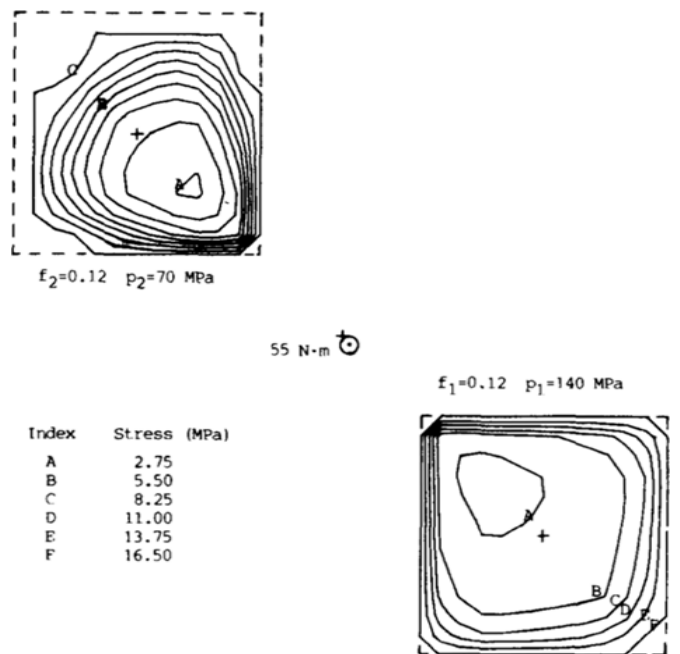

$f_2=0.12$ $p_2=70$ MPa

55 N·m

Index	Stress (MPa)
A	2.75
B	5.50
C	8.25
D	11.00
E	13.75
F	16.50

$f_1=0.12$ $p_1=140$ MPa

Fig. 4   Magnitude of stress on discrete contact area
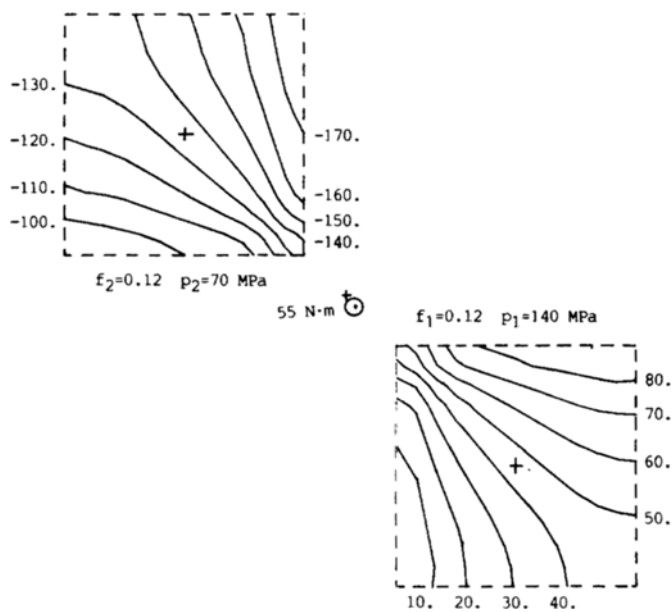
Fig. 5 Direction of stress on discrete contact aree(degree)

$(0.11119 \times 10^{-2} \text{rad})$, with a deviation of 4.30%.

6.2 Disconnected Contact Areas on Semi-Infinite Bodies

Two disconnected square contact areas of the same size(15mm $\times$ 15mm) semi-infinite steel bodies($E = 207\,GPa, \nu = 0.3$) are considered in this case. Uniform pressure is assumed on each contact region($p_1 = 140\,MPa, p_2 = 70\,MPa$). The coefficient of friction on both region 1 and region 2 is assumed to be the same ($f_1 = f_2 = 0.12$). The twisting moment is taken as 55 $N \cdot m$. Since the contact regions are assumed on semi-

Index	Stress (MPa)
A	2.75
B	5.50
C	8.25
D	11.00
E	13.75
F	16.50

Fig. 6 Magnitude of stress on discrete contact area using NLP

Fig. 7 Direction of stress on discrete contact area using NLP(degree)

infinite bodies, the solution for a semi-infinite body subjected to a tangential force is again employed to obtain the influence coefficients(Love, 1944).

The contours of the magnitude and the direction of the traction using the proposed iterative procedure are shown in Figs. 4. and 5. As would be expected, different traction distributions are shown in the two regions, and the center of rotation is consequently found to be displaced from the centroid. Three iterations were necessary and the CPU time elapsed was 2 minutes and 59 seconds on a HARRIS 800.

The contour plots for the corresponding results using the non-linear programming are also shown in Figs. 6 and 7. When the results of the modified linear programming with the centroid as the center of rotation were used as a set of initial guesses for nonlinear programming, the CPU time used was 8 minutes and 56 seconds.

The results using the proposed method compare favorably with those using the non-linear programming.

The location of the center of rotaion with respect to the centroid is plotted versus the applied twisting moment in Fig. 8.
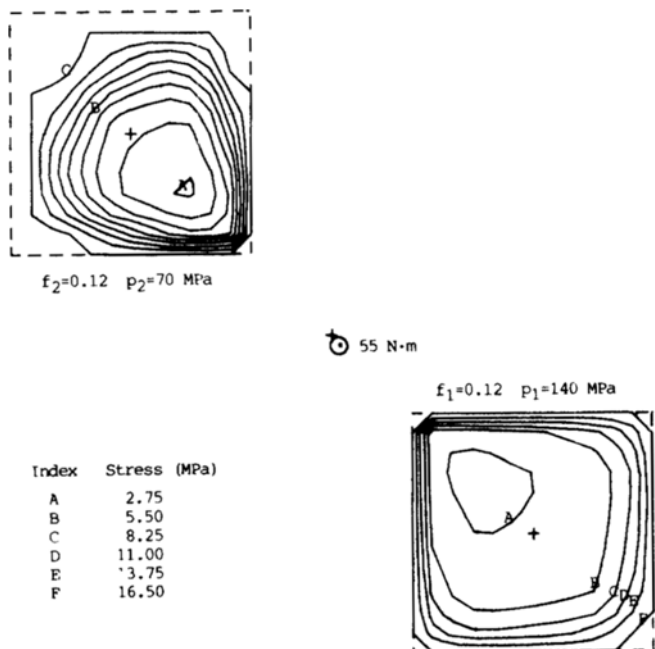
The development of the slip region with the increasing
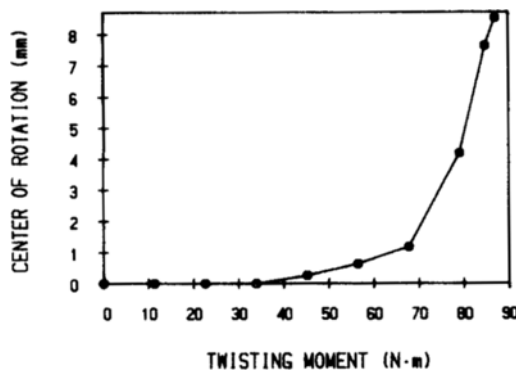
Fig. 8 Location of center rotation w.r.t. centroid

$f_2=0.12$ $p_2=70MPa$

$f_1=0.12$ $p_1=140$ MPa

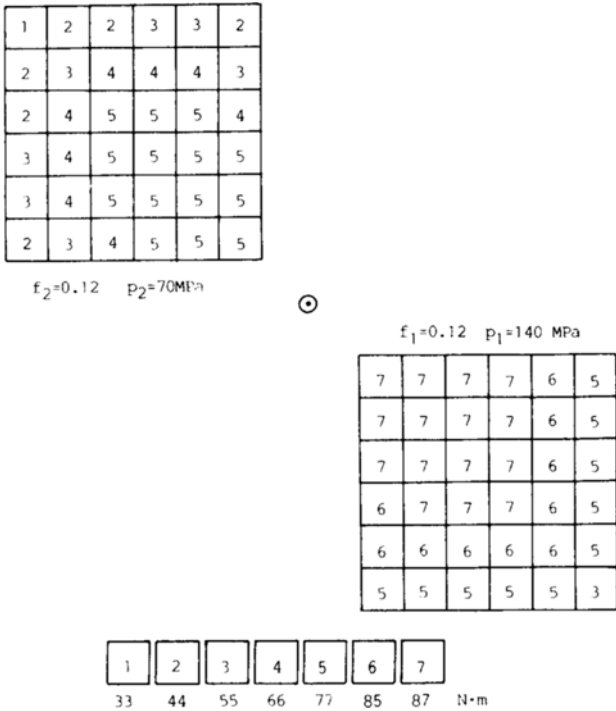| 1	2	3	4	5	6	7
33	44	55	66	77	85	87 N·m

Fig. 9 Development of slip on discrete contact area with increased twisting moment

twisting moment is shown in Fig. 9. The numbers(1,2,3,...,7) refer to the order of the progression of slip when the corresponding twisting moments(33,44,...,87 $N \cdot m$) are increasingly applied.

The curve relating the angle of twist and the twisting moment is plotted in Fig. 10.

7. CONCLUSION

A relatively simple technique has been suggested to solve a frictional contact problem subjected to a twisting moment between two elastic non-conforming bodies. The application of the method to the contact problem between two spheres
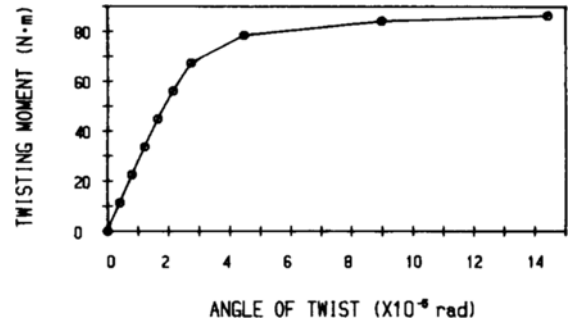

Fig. 10 Twisting moment-angle of twist curve

has proved its accuracy by comparison with Lubkin's theory. The method has been successfully applied to the case of an asymmetric contact area to demonstrate its versatility. The proposed technique seems to be a general and effective procedure for the solution of this class of contact problem.

REFERENCES

Cattaneo, C., 1938, "Sul contatto di due corpi elastici : distribuzione locale degli sforzi", Accademia Lincei, Rendiconti, Series 6, Vol. 27, pp. 342~348, 434~436, 474~478.

Conry, T.F. and Seireg, A.A., 1971, "A Mathematical Programming Method for Design of Elastic Bodies in Contact", J. of Appl. Mech. Vol. 93, pp. 387~392.

Deresiewicz, H., 1952, "Contact of Elastic Spheres under an Oscillating Torsional couple," J. of Appl. Mech., Vol. 76, pp. 52~56.

Hetényi, M. and McDonald, H., 1958, "Contact Stresses under Combined Pressure and Twist", J. of Appl. Mech., Vol. 80, pp. 396~410.

Johnson, K.L., 1985, Contact Mechanics, Cambridge Univ. Press, pp. 92~93.

Love, A.E.H., 1944, A Treatise on the Mathematical Theory of Elasticity, Dover Book Company, Art. 166.

Lubkin, J.L., 1951, "The Torsion of Elastic Spheres in Contact", J. of Appl. Mech., Vol.73, pp. 183~187.

Mindlin, R.D, 1949, "Compliance of Elastic Bodies in Contact," J. of Appl. Mech., Vol. 71, pp. 259~268.